MATH3091: Statistical Modelling II

Problem Sheet 4

1. Suppose $Y \sim \text{Exponential}(\lambda)$, with p.d.f.

$$f_Y(y;\lambda) = \lambda e^{-\lambda y}\,,$$

   for $y > 0$ and $\lambda > 0$. Show that the exponential distribution is a member of the exponential family, and hence find $\mathbb{E}(Y)$, $\text{Var}(Y)$, and the variance function $V(\mu)$.

2. Suppose $Y \sim \text{Geometric}(p)$, with p.d.f.

$$f_Y(y;p) = p(1-p)^{y-1}\,,$$

   for $y \in \{1, 2, 3, \cdots\}$, $p \in (0, 1)$. Show that the geometric distribution is a member of the exponential family, and hence find $\mathbb{E}(Y)$, $\text{Var}(Y)$, and the variance function $V(\mu)$.

3. What is the canonical link function for the exponential distribution (Q1)? What about for the geometric distribution (Q2)?

4. Suppose that Y_i, $i = 1, \cdots, n$, are independent $\text{Poisson}(\mu_i)$ random variables, that x_i is an explanatory variable, and that

$$\log \mu_i = \beta_1 + \beta_2 x_i.$$

   a. Write down the log-likelihood function of β_1 and β_2 explicitly. Hence, derive a pair of simultaneous equations, the solution of which are the maximum likelihood estimates for $\beta = (\beta_1, \beta_2)^\top$.

   b. Express the above model in terms of $\eta = \mathbf{X}\beta$, where $\mathbf{X}$ is the appropriate $n \times 2$ matrix.